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ALTERNATIVE CONDITIONS FOR THE SOLVABILITY OF AN ENCOUNTER-EVASION DIFFERENTIAL GAME*

S. A. VAKHRAMEEV

A positional encounter-evasion differential game with geometric constraints on the players' constraints, depending on the state $\{t, x\}$ of the dynamic system, is examined. It is proved that under specific conditions either the encounter positional game or the evasion positional game is always solvable. The constructions used in the proof are modifications of the extremal construction /1/. A similar problem was examined earlier in /2/ wherein, in particular, a condition of the type of condition (1.3) was suggested.

1. Let the behavior of a controlled system be described by the differential equation

$$x^{\cdot} = f(t, x, u, v), \quad x \in R^n, \quad u \in R^p, \quad v \in R^q \tag{1.1}$$

Here x is the system's phase-coordinate vector, u and v are the controls of the first and second players, respectively. By Ω^k we denote the space of all nonempty compact spaces in R^k with the Hausdorff metric h. Let the mappings

$$P: R \times R^n \to \Omega^p, \quad Q: R \times R^n \to \Omega^q \tag{1.2}$$

be priscribed, satisfying the following conditions:

a) the mappings

$$P(\cdot, x): R \to \Omega^p, \quad Q(\cdot, x): R \to \Omega^q$$

are measurable for all $x \in R^n$ (see /3/);

b) the mappings

$$P(t, \cdot): R^n \to \Omega^n, \quad Q(t, \cdot): R^n \to \Omega^q$$

are continuous for all $t \in R$;

c) measurable mappings

$$P_0: R \to \Omega^p, \quad Q_0: R \to \Omega^q$$

exist such that for all $x \in R^n$

$$P(t, x) \subset P_0(t), \quad Q(t, x) \subset Q_0(t)$$

It is assumed that the function $f: R \times R^n \times R^p \times R^q \to R^n$ on the right-hand side of Eq.(1.1) satisfies the following conditions.

1°. Function $f(t, \cdot, \cdot, \cdot): R^n \times R^p \times R^q \to R^n$ is continuous for all $t \in R$.

2°. Function $f(\cdot, x, u, v): R \to R^n$ is measurable for all $x \in R^n$, $u \in R^p$, $v \in R^q$.

3°. For all $x \in R^n$, $u \in P_0(t)$, $v \in Q_0(t)$

$$|f(t, x, u, v)| \leqslant k(t)(1 + |x|)$$

4°. For all $x, y \in R^n$, $u \in P_0(t)$, $v \in Q_0(t)$

$$|f(t, x, u, v) - f(t, y, u, v)| \leqslant \lambda(t)|x - y|$$

5°. For all $x, y, z \in R^n$

$$\left|\max_{v \in Q(t,x)} \min_{u \in P(t,x)} (z, f(t, x, u, v)) - \min_{u \in P(t,y)} \max_{v \in Q(t,y)} (z, f(t, y, u, v))\right| \leqslant \gamma(t)|z||x - y| \tag{1.3}$$

The functions $k, \lambda, \gamma: R \to R$ are nonnegative and locally Lebesgue-summable. When mappings (1.2) are independent of x condition (1.3) is equivalent to the saddle-point condition in a small game /1/ in the form suggested in /4/. If $\gamma(t) \equiv \gamma_0$, then this condition is equivalent to the condition, proposed in /2/,

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$$a(t, x, z) = \max_{v \in Q(t,x)} \min_{u \in P(t,x)} (z, f(t, x, u, v)) = \min_{u \in P(t,x)} \max_{v \in Q(t,x)} (z, f(t, x, u, v))$$

$$| a(t, x, z) - a(t, y, z) | \leqslant \gamma_0 | x - y |$$

for all $x \in R^n, y \in R^n, z \in R^n, |z| = 1$.

We note two cases when condition (1.3) is fulfilled. Let

$$f(t, x, u, v) = f_1(t, x, u) + f_2(t, x, v)$$

and let mappings (1.2) be such that the sets

$$F_1(t, x) = f_1(t, x, P(t, x)), \; F_2(t, x) = f_2(t, x, Q(t, x))$$

are convex for all $(t, x) \in R^{n+1}$ and

$$h(F_1(t, x), F_1(t, y)) \leqslant \gamma_1(t) | x - y |, \quad h(F_2(t, x), F_2(t, y)) \leqslant \gamma_2(t) | x - y |$$

for all $x, y, \in R^n$, where the functions $\gamma_1, \gamma_2 : R \to R$ are nonnegative and locally summable. In this case inequality (1.3) is fulfilled with the function $\gamma(t) = \gamma_1(t) + \gamma_2(t)$. Now assume that mappings (1.2) satisfy the following Lipschitz conditions:

$$h(P(t, x), P(t, y)) \leqslant \alpha(t) | x - y |, \quad h(Q(t, x), Q(t, y)) \leqslant \beta(t) | x - y |$$

while the function $f: R \times R^n \times R^p \times R^q \to R^n$ on the right-hand part of (1.1) satisfies additionally a Lipschitz condition in the variable u and v $| f(t, x, u_1, v_1) - f(t, x, u_2, v_2) | \leqslant \eta(t) (| u_1 - u_2 | + | v_1 - v_2 |)$ for all $x \in R^n$, $u_1, u_{12} \in P_0(t), v_1, v_2 \in Q_0(t)$. Here the functions $\alpha, \beta, \eta : R \to R$ are nonnegative and locally Lebesgue-summable. Then condition (1.3) is fulfilled with the function $\gamma(t) = \lambda(t) + \eta(t)(\alpha(t) + \beta(t))$ if

$$\max_{v \in Q(t,x)} \min_{u \in P(t,x)} (z, f(t, x, u, v)) = \min_{u \in P(t,x)} \max_{v \in Q(t,x)} (z, f(t, x, u, v))$$

for all $x, z \in R^n$.

For an arbitrary mapping $F: R \times R^n \to \Omega^k$, measurable in the first argument for a fixed second one, we denote by $F(x; t_1, t_2)$ the set of all measurable branches of the mapping $F(\cdot, x): R \to \Omega^k$ onto the half-open interval $t_1 \leqslant t < t_2$. This set is nonempty by the measurable selection theorem /3/.

A mapping $U \div U(t, x)$ which associates a nonempty set from $P(x; t, \infty)$ with an arbitrary position $(t, x) \in R^{n+1}$ is called a strategy of the first player. A strategy $V \div V(t, x)$ of the second player is defined analogously. Suppose that the first player has chosen a strategy $U \div U(t, x)$. Consider the partitioning Δ of the semiaxis $[t_0, \infty)$ into a system of half-open intervals

$$\tau_i \leqslant t < \tau_{i+1}, \quad \tau_0 = t_0, \quad \tau_i \to \infty, \quad i \to \infty$$

Let $| \Delta | = \sup_i (\tau_{i+1} - \tau_i)$. A solution of the differential equation

$$x_\Delta^{\cdot} = f(t, x_\Delta, u_i(t), v_i(t)), \quad \tau_i \leqslant t < \tau_{i+1}$$

$$u_i(\cdot) \in U(\tau_i, x_\Delta(\tau_i)), \; v_i(\cdot) \in Q(x_\Delta(\tau_i); \tau_i, \tau_{i+1}), \quad i = 0, 1, \ldots, \quad x_\Delta(t_0) = x_0$$

is called an Euler polygonal line generated by strategy $U \div U(t, x)$. It can be shown that every Euler polygonal line $x_\Delta(t) = x_\Delta(t; t_0, x_0, U, v)$ satisfies the differential inclusion

$$x^{\cdot} \in \operatorname{conv} f(t, x, P_0(t), Q_0(t)) \tag{1.4}$$

Since the set of solutions of this differential inclusion with the initial condition $x(t_0) \in X_0$, where $X_0 \in \Omega^n$ is compact in $C_n[t_0, t_1]$, the following definition is correct.

A function $x(t) = x(t; t_0, x_0, U)$ for which we can find a sequence $x_{\Delta_k}(t) = x_{\Delta_k}(t; t_0, x_0^k, U, v_k)$ of Euler polygonal lines on any finite interval $t_0 \leqslant t \leqslant t_1$, such that

$$x_{\Delta_k}(t) \rightrightarrows x(t), \quad t_0 \leqslant t \leqslant t_1$$

as $x_0^k \to x_0$, $| \Delta_k | \to 0$, $k \to \infty$, is called a motion generated by the first player's strategy $U \div U(t, x)$. A motion $x(t) = x(t; t_0, x_0, V)$ generated by the second player's strategy $V \div V(t, x)$ is defined analogously. We note that every motion of the first and second player, starting from point x_0 at instant t_0, is a solution of the differential inclusion

$$x^{\cdot} \in \operatorname{conv} f(t, x, P(t, x), Q(t, x)), \quad x(t_0) = x_0$$

Further, we note that the estimate

$$\max_{v \in Q_0(t)} \max_{u \in P_0(t)} | f(t, x(t), u, v) | \leqslant m_G(t) \tag{1.5}$$

where the function $m_G: R \to R$ is nonnegative and locally Lebesgue-summable and depends only on G, is valid for an arbitrary solution $x(t)$, $t \in R$, of differential inclusion (1.4) with initial conditions $(t_0, x(t_0)) \in G$, where $G \in \Omega^{n+1}$.

Let nonempty closed sets M and N in the position space R^{n+1}, an initial position (t_0, x_0), and an instant $T \geqslant t_0$ be specified. The encounter-evasion game being examined consists of the following two problems.

Problem 1. Find a strategy $U^* \div U^*(t, x)$ which ensures the contact

$$(t, x(t)) \in N,\ t_0 \leqslant t < \tau,\ (\tau, x(\tau)) \in M, \quad \tau \leqslant T$$

for all motions $x(t) = x(t; t_0, x_0, U^*)$.

Problem 2. Find open neighborhoods $H(N)$ and $G(M)$ of sets N and M and a strategy $V^* \div V^*(t, x)$, which exclude the contact

$$(t, x(t)) \in H(N), \quad t_0 \leqslant t < \tau, \quad (\tau, x(\tau)) \in G(M), \quad \tau \leqslant T$$

for all motions $x(t) = x(t; t_0, x_0, V^*)$.

2. We say that a set $W \subset R^{n+1}$ is u-stable if for any position $(t_*, x_*) \in W$, instant $t^* > t_*$, and control $v^*(\cdot) \in Q(x_*; t_*, t^*)$ we can find a solution $x(t)$, $t_* \leqslant t \leqslant t^*$, of differential inclusion

$$x^{\cdot}(t) \in \operatorname{conv} f(t, x(t), P(t, x_*), v^*(t)), \quad x(t_*) = x_*$$

such that $(t^*, x(t^*)) \in W$ or $(\tau, x(\tau)) \in M$ for some τ, $t_* \leqslant \tau \leqslant t^*$. We say that a set $W \subset R^{n+1}$ is v-stable if for any position $(t_*, x_*) \in W$, instant $t^* > t_*$, and control $u^*(\cdot) \in P(x_*; t_*, t^*)$ we can find a solution $x(t)$, $t_* \leqslant t \leqslant t^*$, of differential inclusion

$$x^{\cdot}(t) \in \operatorname{conv} f(t, x(t), u^*(t), Q(t, x_*)), \quad x(t_*) = x_*$$

such that $(t^*, x(t^*)) \in W$ or $(\tau, x(\tau)) \notin H(N)$ for some τ, $t_* \leqslant \tau \leqslant t^*$. It can be prove that the property of u-stability (v-stability) in invariant relative to the closure operation, i.e., if set W is u-stable (v-stable), then its closure $\operatorname{cl} W$ is u-stable (v-stable).

We present an example of a u-stable set. Let

$$f(t, x, u, v) = A(t)x + u - v$$

and let mappings (1.2) be independent of x: $P(t, x) \equiv P(t)$, $Q(t, x) = Q(t)$, and be locally Lebesgue-integrable (see /5/). Let $X(t, t_0)$ be the fundamental matrix of solution of the homogeneous equation

$$x^{\cdot} = A(t)x$$

Assume that matrix $A(t)$ is locally Lebesgue-integrable. Further, we assume that set N coincides with R^{n+1} and that set $M = R \times M_0$, where the set $M_0 \subset R^n$ is nonempty and closed. By $A \stackrel{*}{-} B$ we denote the geometric difference of sets A and B from R^n

$$A \stackrel{*}{-} B = \{z \in R^n \mid z + B \subset A\}$$

The u-stability of set

$$W = \left\{(t, x) \in R^{n+1} \mid X(T, t)x \in M_0 - \int_t^T \{X(T, \tau)P(\tau) \stackrel{*}{-} X(T, \tau)Q(\tau)\}\, d\tau,\ t \leqslant T\right\}$$

can be verified directly.

A strategy $U^e \div U^e(t, x)$ of the first player, extremal to the closed set $W \subset R^{n+1}$, is defined as follows. Let (t_*, x_*) be an arbitrary position, $\Gamma_{t_*} = \{(t, x) \in R^{n+1} \mid t = t_*\}$. If $W \cap \Gamma_{t_*} = \varnothing$, then we assume $U^e(t_*, x_*) = P(x_*; t_*, \infty)$; if $W \cap \Gamma_{t_*} \neq \varnothing$, then we assume

$$U^e(t_*, x_*) = \{u^*(\cdot) \in P(x_*; t_*, \infty) \mid \max_{v \in Q(t, x_*)} (x_* - w_*, f(t, x_*, u^*(t), v)) =$$
$$\min_{u \in P(t, x_*)} \max_{v \in Q(t, x_*)} (x_* - w_*, f(t, x_*, u, v)), \quad t \geqslant t_*\}$$

where w_* is the vector of the section of set W by the hyperplane Γ_{t_*}, which is closest to position (t_*, x_*). A strategy $V^e \div V^e(t_*, x_*)$ of the second player, extremal to the closed set $W \subset R^{n+1}$, is defined as follows. If $\Gamma_{t_*} \cap W = \varnothing$, then $V^e(t_*, x_*) = Q(x_*; t_*, t^*)$. If $\Gamma_{t_*} \cap W \neq \varnothing$, then

$$V^e(t_*, x_*) = \{v^*(\cdot) \in Q(x_*; t_*, \infty) \mid \min_{u \in P(t, x_*)} (w_* - x_*, f(t, x_*, u, v^*(t))) =$$
$$\max_{v \in Q(t, x_*)} \min_{u \in P(t, x_*)} (w_* - x_*, f(t, x_*, u, v)), t \geqslant t_*\}$$

3. Let the function $x(t), t \in R$, satisfy the equation

$$\dot{x} = f(t, x, u^*(t), v(t)), \quad x(t_*) = x_*$$

while the function $y(t), t \in R$, satisfies the differential inclusion

$$\dot{y} \in \operatorname{conv} f(t, y, P(t, y_*), v^*(t)), \quad y(t_*) = y_*$$

where the function $v(\cdot) \in Q(x_*; t_*, \infty)$ is arbitrary, while the functions $u^*(\cdot) \in P(x_*; t_*, \infty)$ and $v^*(\cdot) \in Q(y_*; t_*, \infty)$ are chosen from the conditions

$$\max_{v \in Q(t, x_*)} (x_* - y_*, f(t, x_*, u^*(t), v)) = \min_{u \in P(t, x_*)} \max_{v \in Q(t, x_*)} (x_* - y_*, f(t, x_*, u, v))$$

$$\min_{u \in P(t, y_*)} (x_* - y_*, f(t, y_*, u, v^*(t))) = \max_{v \in Q(t, y_*)} \min_{u \in P(t, y_*)} (x_* - y_*, f(t, y_*, u, v)), \quad t \geqslant t_*$$

Let $\rho(t) = |x(t) - y(t)|, \ t \geqslant t_*$. It turns out that the estimate

$$\rho^2(t) \leqslant \rho^2(t_*)\left(1 + 2\int_{t_*}^{t} \gamma(\tau)\, d\tau\right) + \int_{t_*}^{t} \varphi(\tau, t_*)\, m(\tau)\, d\tau \qquad (3.1)$$

$$t \geqslant t_*$$

where

$$m(t) = 4g\lambda(t) + 8m_G(t), \quad g = \operatorname{diam} G, \quad \varphi(t, t_*) = \int_{t_*}^{t} m_G(\tau)\, d\tau$$

and the function $m_G(\cdot)$ is from (1.5), holds for all positions (t_*, x_*) and (t_*, y_*) from the compact space $G \in \Omega^{n+1}$.

The following statements are obtained, analogously to /1/, with the aid of this estimate.

Lemma 1. If $W \subset R^{n+1}$ is a closed u-stable set, $U^e \div U^e(t, x)$ is a strategy extremal to this set, and $(t_0, x_0) \in W$, then the inclusion $(t, x(t)) \in W$ is fulfilled for any motion $x(t) = x(t; t_0, x_0, U^e)$ up to contact $(\tau, x(\tau)) \in M$. If contact with M does not occur at all for some motion $x(t) = x(t; t_0, x_0, U^e)$, then for such a motion $(t, x(t)) \in W$ for all $t \geqslant t_0$.

Lemma 2. Let $W \subset R^{n+1}$ be a closed v-stable set, $V^e \div V^e(t, x)$ be a strategy extremal to this set, and $(t_0, x_0) \in W$. Then $(t, x(t)) \in W$ for any motion $x(t) = x(t; t_0, x_0, V^e)$, up to the instant τ when $(\tau, x(\tau)) \notin H(N)$. If $(t, x(t)) \in H(N)$ for all time for some motion $x(t) = x(t; t_0, x_0, V^e)$, then $(t, x(t)) \in W$ for all $t \geqslant t_0$.

From these statements follows

Theorem. Suppose that an initial position $(t_0, x_0) \in R^{n+1}$ has been given and that an instant $T \geqslant t_0$ has been chosen. When all the conditions formulated in Sect.1 are fulfilled, either Problem 1 or Problem 2 is always solvable.

In contrast to /2/ we assume the measurability in t of mappings (1.2). The latter, in particular, required the adoption of other definitions of the basic elements of the game, such as strategies, stable sets, etc., in comparison with /1,2/. This, in its own turn, compelled us to use the estimate (3.1), different from those in /1,2/, to prove the barrier properties of the extremal strategies, and to prove the invariance of the property of $u(v)$-stability relative to the closure operation. Finally, the remaining conditions imposed on the game in this article are somewhat weaker than the conditions in /1,2/.

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